

MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS INVOLVING MULTIDIMENSIONAL LAPLACE TRANSFORM

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Abstract: In the present paper the certain multidimensional fractional integral operators involving a general class of polynomials for multidimensional Laplace transform are derived. The fractional integral operators studied here are general in nature, as by taking suitable values of the coefficients involved the general class of polynomials can be reduced to the classical orthogonal polynomials, the Bessel polynomials, and several other classes of generalized hypergeometric polynomials

Keywords: Fractional operators, general class of polynomial, multidimensional Laplace transform.

I. INTRODUCTION

Due to the important role played by the fractional integral operators in several problems of mathematical physics and applied mathematics, various generalization of these operators are introduced and studied by several authors. A detailed study of these operators, their properties and applications was carried out by Miller and Ross [1], Samko et.al. [2], Debnath and Bhatta [3] and, Kilbas et.al. [4]. Ross [5] obtained the fractional integral formulae with the help of series expansion method. Banerji and Choudhary [6], Raina and Ladda [7], Raina [8] and, Miyakoda [9] have studied and defined fractional calculus operators using series expansion method. The use of two dimensional Laplace transform is evident in the solutions of various types of partial differential equations, difference equations, in the study of useful properties of various special functions. Laplace and Fourier transforms in

three and more variables has also inspired the development and applications of operational methods. This paper attempt to obtain certain fractional calculus formulae, involving multidimensional fractional integral operators and multidimensional Laplace Transform.

II. Definition

Some Definitions:- In this section definitions of the functions under consideration are given.

2.1 General Class of Polynomials

A general class of polynomials introduced by Srivastava [10] is

$$S_v^U[x] = \sum_{j=0}^{\lfloor U \rfloor} \frac{(-v)_{Uj} A_{v,j}}{j!} x^j$$

... (1.1) where U is an arbitrary positive integer and the coefficients $A_{v,j} (v, j \geq 0)$ are arbitrary constants, real or complex.

2.2 Multi-dimensional Fractional Integral Operators

Fractional integral operators defined in [11] are

$$\begin{aligned} & \mathcal{J}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} f(x_1, \dots, x_r) \\ &= \prod_{i=1}^r \frac{x_i^{-h_i - \mu_i}}{\Gamma \mu_i} \int_0^{x_i} \dots \int_0^{x_r} \prod_{i=1}^r (x_i - t_i)^{-\mu_i - 1} t_i^{h_i} \\ & \cdot S_n^m \left[z \prod_{i=1}^r \left(1 - \frac{t_i}{x_i} \right)^{\rho_i} \right] f(t_1, \dots, t_r) dt_1 \dots dt_r \quad \dots(1.2) \end{aligned}$$

where

$$\operatorname{Re}(\mu_i + \rho_j) > 0, (i = 1, \dots, r; j = 0, 1, 2, \dots, [n/m])$$

And

$$\begin{aligned} & \mathcal{K}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} f(x_1, \dots, x_r) \\ &= \prod_{i=1}^r \frac{x_i^{h_i}}{\Gamma \mu_i} \int_{x_i}^{\infty} \dots \int_{x_r}^{\infty} \prod_{i=1}^r (t_i - x_i)^{\mu_i - 1} t_i^{-h_i - \mu_i} \\ & \cdot S_n^m \left[z \prod_{i=1}^r \left(1 - \frac{x_i}{t_i} \right)^{\rho_i} \right] f(t_1, \dots, t_r) dt_1 \dots dt_r \quad \dots(1.3) \end{aligned}$$

where

$$\operatorname{Re}(\mu_i) > -\operatorname{Re}(\rho_j), (i = 1, \dots, r; j = 0, 1, 2, \dots, [n/m])$$

The fractional integral operators defined by equations (1.2) and (1.3) can be considered as the generalizations of the repeated forms of the corresponding one dimensional operators defined by Erdelyi [12] and Kober [13].

2.3 Multi-dimensional Laplace Transform

The multidimensional Laplace transform of the function $f(x_1, \dots, x_r)$ is defined as

$$\begin{aligned} (\mathcal{L}f)s &= \mathcal{L}\{f(x_1, \dots, x_r); s_1, \dots, s_r\} \\ &= \int_0^{\infty} \dots \int_0^{\infty} \exp\left(-\sum_{i=1}^r x_i s_i\right) f(x_1, \dots, x_r) dx_1 \dots dx_r \end{aligned}$$

Where $\Re(s_i) > 0, i = 1, \dots, r$

2.4 Mellin Transform

The multidimensional Mellin transform [14] of the function $f(x_1, \dots, x_r)$ is defined as

$$(\mathfrak{M}f)s = \int_0^{\infty} \dots \int_0^{\infty} x_1^{s_1 - 1} \dots x_r^{s_r - 1} f(x_1, \dots, x_r) dx_1 \dots dx_r$$

Where $x_1, \dots, x_r > 0$

Main Results

Theorem 1

Let $f(t_1, \dots, t_r)$ be such that the multidimensional Laplace transform represented as

$$g(s_1, \dots, s_r) = \mathcal{L}\{f(t_1, \dots, t_r); s_1, \dots, s_r\}$$

exists then for $\Re(s_i) > 0, i = 1, \dots, r$

$$\mathcal{J}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} [g_1(s_1, \dots, s_r)]$$

$$\begin{aligned} &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma \mu_i} \\ & \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i + j + 1)}{\Gamma(h_i + j + \mu_i + \rho_i k + 1)} \mathfrak{m}[f(j+1)] \end{aligned}$$

Proof:

$$\begin{aligned}
 \text{Let } \Delta_1 &= \mathcal{J}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} [g_1(s_1, \dots, s_r)] \\
 &= s_1^{-h_1 - \mu_1} \dots s_r^{-h_r - \mu_r} s_1^{h_1} \dots s_r^{h_r} \\
 &\cdot \frac{1}{\Gamma \mu_1 \dots \Gamma \mu_r} \int_0^{s_1} \dots \int_0^{s_r} (s_1 - t_1)^{\mu_1 - 1} \dots (s_r - t_r)^{\mu_r - 1} \\
 &\cdot S_n^m \left[z \left(1 - \frac{t_1}{s_1} \right)^{\rho_1} \dots \left(1 - \frac{t_r}{s_r} \right)^{\rho_r} \right] t_1^{h_1} \dots t_r^{h_r} g(t_1 \dots t_r) dt_1 \dots dt_r \\
 &= \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i}}{\Gamma \mu_i} \int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i - 1} \\
 &\cdot S_n^m \left[z \prod_{i=1}^r \left(1 - \frac{t_i}{s_i} \right)^{\rho_i} \right] g_1(t_1, \dots, t_r) dt_1 \dots dt_r
 \end{aligned}$$

$$\begin{aligned}
 &= \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i}}{\Gamma \mu_i} \int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i - 1} \\
 &\cdot \left[\sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} (-n)_{mk} A_{n,k} \frac{z^k \prod_{i=1}^r \left(1 - \frac{t_i}{s_i} \right)^{\rho_i k}}{k!} \right] \\
 &\cdot \left[\int_0^\infty \dots \int_0^\infty \exp \left(- \sum_{i=1}^r x_i t_i \right) f(x_1, \dots, x_r) dx_1 \dots dx_r \right] \cdot dt_1 \dots dt_r
 \end{aligned}$$

where $\text{Re}(x_i) > 0$

$$\begin{aligned}
 \Delta_1 &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r) \\
 &\cdot \left[\int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i + \rho_i k - 1} e^{-x_i t_i} dt_1 \dots dt_r \right] dx_1 \dots dx_r
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \left[\int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r) \right. \\
 &\cdot \left. \int_0^{s_1} \dots \int_0^{s_r} \prod_{i=1}^r t_i^{h_i} (s_i - t_i)^{\mu_i + \rho_i k - 1} \sum_{j=0}^\infty \frac{(-x_i t_i)^j}{j!} dt_1 \dots dt_r \right] dx_1 \dots dx_r \\
 &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \left[\int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r) \right. \\
 &\left. \prod_{i=1}^r \sum_{j=0}^\infty \frac{(-x_i)^j}{j!} \int_0^{s_1} \dots \int_0^{s_r} t_i^{h_i + j} (s_i - t_i)^{\mu_i + \rho_i k - 1} dt_1 \dots dt_r \right] dx_1 \dots dx_r
 \end{aligned}$$

Using $\int_0^u x^{\nu-1} (u-x)^{\mu-1} dx = u^{\mu+\nu-1} B(\nu, \mu)$ [15]

$$\begin{aligned}
 \Delta_1 &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{-h_i - \mu_i - \rho_i k}}{\Gamma \mu_i} \\
 &\cdot \left[\int_0^\infty \dots \int_0^\infty f(x_1, \dots, x_r) \sum_{j=0}^\infty \frac{(-x_i)^j}{j!} s_i^{h_i + j + \mu_i + \rho_i k} \right] \\
 &\cdot B(h_i + j + 1, \mu_i + \rho_i k) \\
 &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{1}{\Gamma(\mu_i)} \sum_{j=0}^\infty \frac{(-1)^j s_i^j}{j!} \\
 &\frac{\Gamma(h_i + j + 1) \Gamma(\mu_i + \rho_i k)}{\Gamma(h_i + j + \mu_i + \rho_i k + 1)} \int_0^\infty \dots \int_0^\infty x_i^j f(x_1, \dots, x_r) dx_1 \dots dx_r
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma \mu_i} \\
 &\sum_{j=0}^\infty \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i + j + 1)}{\Gamma(h_i + j + \mu_i + \rho_i k + 1)} \mathbf{m}[f : (j+1)]
 \end{aligned}$$

Theorem 2

Let $f(t_1, \dots, t_r)$ be such that the multidimensional Laplace transform represented as

$$g(s_1, \dots, s_r) = \mathcal{L}\{f(t_1, \dots, t_r); s_1, \dots, s_r\}$$

exists then for $\Re(s_i) > 0, i = 1, \dots, r$

$$\mathcal{K}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} g(s_1, \dots, s_r)$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma(\mu_i)}$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i - j)}{\Gamma(h_i - j + \mu_i + \rho_i k)} \mathfrak{m}[f : (j+1)] \Re(h_i + \mu_i + \rho_i k - j) > \Re(\mu_i + \rho_i k) > 0$$

Proof:

$$\text{Let } \Delta_2 = \mathcal{K}_{\mu_1, \dots, \mu_r}^{z; m, n; \rho, h} g(s_1, \dots, s_r)$$

$$= \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma(\mu_i)} \int_{s_i}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{-h_i - \mu_i} (t_i - s_i)^{\mu_i - 1}$$

$$\cdot \left[\sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} (-n)_{mk} A_{n,k} \frac{z^k \prod_{i=1}^r \left(1 - \frac{s_i}{t_i}\right)^{\rho_i k}}{k!} \right]$$

$$\cdot \left[\int_0^{\infty} \dots \int_0^{\infty} \exp\left(-\sum_{i=1}^r x_i t_i\right) f(x_1, \dots, x_r) dx_1 \dots dx_r \right] dt_1 \dots dt_r$$

$$\begin{aligned} &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma(\mu_i)} \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \\ &\cdot \left[\int_{s_1}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{-h_i - \mu_i - \rho_i k} (t_i - s_i)^{\mu_i + \rho_i k - 1} e^{-x_i t_i} dt_1 \dots dt_r \right] dx_1 \dots dx_r \\ &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma(\mu_i)} \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \\ &\cdot \left[\int_{s_1}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{-h_i - \mu_i - \rho_i k} (t_i - s_i)^{\mu_i + \rho_i k - 1} \sum_{j=0}^{\infty} \frac{(-x_i t_i)^j}{j!} dt_1 \dots dt_r \right] dx_1 \dots dx_r \\ &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma(\mu_i)} \left[\int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \sum_{j=0}^{\infty} \frac{(-x_i)^j}{j!} \right. \\ &\left. \int_{s_1}^{\infty} \dots \int_{s_r}^{\infty} \prod_{i=1}^r t_i^{j - h_i - \mu_i - \rho_i k} (t_i - s_i)^{\mu_i + \rho_i k - 1} dt_1 \dots dt_r \right] dx_1 \dots dx_r \end{aligned}$$

where

$$\text{Using } \int_u^{\infty} x^{-\nu} (x-u)^{\mu-1} dx = u^{\mu-\nu} B(\nu-\mu, \mu) \quad [15]$$

$$\Delta_2 = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{s_i^{h_i}}{\Gamma(\mu_i)} \left[\int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_r) \sum_{j=0}^{\infty} \frac{(-x_i)^j}{j!} s_i^{h_i} B(h_i - j, \mu_i + \rho_i k), \right]$$

$$\begin{aligned} &= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{1}{\Gamma(\mu_i)} \sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i - j) \Gamma(\mu_i + \rho_i k)}{\Gamma(h_i - j + \mu_i + \rho_i k)} \\ &\cdot \int_0^{\infty} \dots \int_0^{\infty} x_i^j f(x_1, \dots, x_r) dx_1 \dots dx_r \end{aligned}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-n)_{mk} A_{n,k} z^k}{k!} \prod_{i=1}^r \frac{\Gamma(\mu_i + \rho_i k)}{\Gamma(\mu_i)}$$

$$\sum_{j=0}^{\infty} \frac{(-1)^j s_i^j}{j!} \frac{\Gamma(h_i - j)}{\Gamma(h_i - j + \mu_i + \rho_i k)} \mathfrak{m}[f : (j+1)]$$

III. CONCLUSION

In this paper two theorems on certain multidimensional fractional integral operators involving a general class of polynomials for multidimensional Laplace transform are derived. The fractional integral operators studied here are general in nature, since by taking suitable values of the coefficients involved the general class of polynomials can be reduced to the classical orthogonal polynomials, the Bessel polynomials, and several other classes of generalized hypergeometric polynomials. These results can be further extended by special choices of multidimensional transforms. The one and two-dimensional analogous of Theorems 1 and 2 can be deduced easily. The results derived in this section are general in nature and will find useful applications in the theory of fractional calculus and integral transform

IV. REFERENCES

- [1] Miller, K.S. and Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, New York: John Wiley and Sons, 1993.
- [2] Samko, S.G., Kilbas, A.A. and Marichev, O.I., *Fractional Integrals and Derivatives of Fractional Order and Some of Their Applications*, New York: Gordon and Breach, 1993.
- [3] Debnath, L. and Bhatta, D., *Integral Transforms and Their Applications*, Boca Raton Florida: Chapman and Hall/CRC Press, 2006.
- [4] Kilbas A.A., Srivastava H.M. and Trujillo J.J., *Theory and Applications of Fractional Differential Equations*, Amsterdam, etc: Elsevier, North Holland Math. Studies, 2006.
- [5] Ross, B., "A Formula for the Fractional Integration and Differentiation of $(a + bx)^c$," *J. Fractional Calculus*, vol. 5, pp. 87-90, 1994.
- [6] Banerji, P.K., Choudhary, S., "On the Fractional Calculus of a general class of polynomials," *Indian J. Pure Appl. Math.*, vol. 27, no. 7, pp. 675-679, 1996.
- [7] Raina, R.K. and Ladda, R.K., "A new family of functional series relations involving digamma functions," *Annales mathematiques Blaise Pascal*, vol. 3, pp. 189-198, 1996.
- [8] Raina, R.K., "A series relation by means of certain fractional calculus operators," *Indian J. Pure Appl. Math.*, vol. 21, no. 2, pp. 172-175, 1990.
- [9] Miyakoda, T., "Discretized Fractional Calculus with a Series of Chebyshev Polynomial, Elec," *Electronic Notes in Theoretical Computer Science (ENTCS)* 225, (2009), 239-244, vol. 225, pp. 239-244, 2009.
- [10] Srivastava H. M., "A contour integral involving Fox's H-function," *Indian J. Math.*, vol. 14, pp. 1-6, 1972.
- [11] Srivastava H. M., Goyal S. P., Jain R. M., "Fractional integral Operators Involving a General Class of Polynomials," *Journal of Mathematical Analysis and Applications*, vol. 148, pp. 87-100, 1990.
- [12] Erdlyi A., "Fractional integrals of generalized functions," *Fractional Calculus and Its Applications* (B. Ross, Ed.), Vols. Springer-Verlag, Berlin/Heidelberg/New York, pp. 151-170, 1975.
- [13] Kober H., "On fractional integrals and derivatives," *Quart. J. Math. (Oxford)*, vol. 11, no. 43, pp. 193-211, 1940.
- [14] Samko S.G., Kilbas A.A. and Marichev O.I., *Fractional Integrals and Derivatives: Theory and Applications*, Yverdon, Switzerland. : Gordon and Breach, 1993.
- [15] Gradshteyn I.S., *Table of Integrals, series, and products*, Academic Press, 1980.